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Period doubling in six-dimensional symmetric volume-preserving maps

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Abstract. Period doubling in three symmetrically coupled two-dimensional area-preserving maps is numerically studied. It is found that there is a two-dimensional manifold on which all period-2^n ($n = 0, 1, 2, \ldots$) orbits of a period-doubling bifurcation sequence lie. On this manifold, the universality classes of period doubling are just the three classes for two symmetrically coupled two-dimensional area-preserving maps, each characterised by its own Feigenbaum constants, reported by Mao and Helleman. It is also reported that, for three non-symmetrically coupled two-dimensional area-preserving maps, three maps which are fixed under the renormalisation operator have been found. The relevant eigenvalues of perturbation around the fixed maps are again the same as those found for two non-symmetrically coupled two-dimensional area-preserving maps, by Mao and Greene. The above-mentioned numerical and renormalisation results for the six-dimensional maps agree with each other.

1. Introduction

A transition from periodic motion to chaotic motion in Hamiltonian systems with N degrees of freedom can be studied via the period-doubling route of volume-preserving $(2N-2)$-dimensional maps. For $N=2$, period doubling in two-dimensional (2D) area-preserving maps, such as the Hénon map

$$x' = -y + f(x) \qquad y' = x \tag{1.1}$$

has been extensively studied [1-4]. For $N=3$, period doubling in four-dimensional (4D) volume-preserving maps has attracted more and more interest in recent years [5-9]. Using the following symmetric 4D volume-preserving maps, i.e. two symmetrically coupled two-dimensional area-preserving maps,

$$\begin{aligned} x_1' &= -y_1 + f(x_1, x_2) \qquad & y_1' &= x_1 \\ x_2' &= -y_2 + f(x_2, x_1) \qquad & y_2' &= x_2 \end{aligned} \tag{1.2}$$

period doubling has been studied numerically [8] and by a renormalisation calculation [9]. In the numerical study, the 4D maps have two parameters, and thus there are two convergence rates of the parameters (i.e. Feigenbaum constants), δ_1 and δ_2. It was found that $\delta_1 = 8.721\ldots$ (the same as 2D area-preserving maps) in all cases and $\delta_2 = -4.404\ldots$, $4.000\ldots$ and $-2.000\ldots$ for almost all bifurcation paths in the three

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kinds of universality (E, L and U) classes respectively. For the exceptional paths in the L and U classes, $\delta_2 = -15.1\ldots$. In the renormalisation study for two non-symmetrically coupled 2D area-preserving maps, three maps which are fixed under the renormalisation operator have been determined and called the E, L and U maps. The relevant eigenvalues of the perturbation around the fixed maps are $\delta_2 = -4.4\ldots$, 4 and -2 respectively for the E, L and U maps, and $\delta_1 = 8.721\ldots$ for all three maps. These δ_1 and δ_2 values agree with the numerical results.

In this paper, we will study period doubling in symmetric six-dimensional (6D) volume-preserving maps ($N = 4$),

$$\begin{aligned} x_1' &= -y_1 + f(x_1, x_2, x_3) \qquad & y_1' &= x_1 \\ x_2' &= -y_2 + f(x_2, x_3, x_1) \qquad & y_2' &= x_2 \\ x_3' &= -y_3 + f(x_3, x_1, x_2) \qquad & y_3' &= x_3 \end{aligned} \tag{1.3}$$

i.e. three symmetrically coupled 2D area-preserving maps with

$$f(x_1, x_2, x_3) = f(x_1, x_3, x_2). \tag{1.4}$$

Here $f(x_1, x_2, x_3)$ is a quadratic polynomial. Two or more parameters could be introduced into our 6D map. However, only two parameters have been actually used in the map (see equation (3.1)) in the numerical study because the renormalisation work (see § 4) shows that there are only two relevant directions at each fixed point of the renormalisation operator in function space. The 6D map (1.3) has (see § 2) several lower-dimensional manifolds on each of which lie all period-2^n ($n = 0, 1, 2, \ldots$) orbits of the corresponding bifurcation sequence. One such manifold is $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, a 2D manifold, denoted by M_0. On this manifold M_0, period-doubling sequences have been obtained (see § 3). These are the same sequences for the 4D map (1.2), with $x_3 = x_2$ and $y_3 = y_2$ here for our 6D map (1.3). Hence the same three universality classes (characterised by the same Feigenbaum constants δ_1 and δ_2) as in the 4D case have been found. In § 4, we perform a renormalisation calculation for three non-symmetrically coupled 2D area-preserving maps, which is of course valid for the symmetric case as well. Three maps which are fixed points of the renormalisation operator have been found. Around each fixed map, the eigenvalues of perturbation in the parameter space have been found to be identical to those for the two non-symmetrically coupled 2D area-preserving maps.

I would like to emphasise here that the 6D maps studied in this paper are in a special class of 6D maps. Probably, more universality classes of period doubling in a generic 6D map are waiting to be discovered.

2. Symmetries in 6D symmetric volume-preserving maps

Consider the 6D map (1.3). It can be rewritten in the following vector form:

$$\boldsymbol{x} = -\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{x}) \qquad \boldsymbol{y} = \boldsymbol{x} \tag{2.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are 3-vectors, and $\boldsymbol{f}$ a non-linear transformation, i.e.

$$\begin{aligned} \boldsymbol{x} &\equiv (x_1, x_2, x_3) \\ \boldsymbol{y} &\equiv (y_1, y_2, y_3) \\ \boldsymbol{f}(\boldsymbol{x}) &\equiv (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)). \end{aligned} \tag{2.2}$$

The symmetry of coupling (i.e. the three non-linear functions in map (1.3) are the same, with only their variables interchanged) takes the following form:

$$f_1(x_1, x_2, x_3) = f_2(x_3, x_1, x_2) = f_3(x_2, x_3, x_1). \tag{2.3}$$

The additional symmetry, equation (1.4), becomes

$$f_1(x_1, x_2, x_3) = f_1(x_1, x_3, x_2). \tag{2.4}$$

As can be seen, our map is very special. However, a numerical study (see § 3) of period doubling in this special map gives all universality classes found in renormalisation calculation (see § 4) for map (2.1) without those restrictions such as symmetries of equations (2.3) and (2.4).

2.1. Symmetries

An operator S is called a symmetry of a map T if S and (TS) are involutions [1, 9, 10], i.e.

$$S^2 = 1 \tag{2.5}$$

$$(TS)^2 = 1. \tag{2.6}$$

From this definition, we have

$$STS = T^{-1} \tag{2.7}$$

$$ST^{-1}S = T. \tag{2.8}$$

The symmetry has a property that if $\boldsymbol{r} = (x_1, y_1, x_2, y_2, x_3, y_3)$ is a fixed point of T^N,

$$T^N\boldsymbol{r} = \boldsymbol{r} \tag{2.9}$$

then $(S\boldsymbol{r})$ is also a fixed point of T^N,

$$T^N(S\boldsymbol{r}) = S\boldsymbol{r}. \tag{2.10}$$

In fact, according to (2.8), $\boldsymbol{r}$ is also a fixed point of $(T^{-1})^N$ which can be rewritten, due to equations (2.8) and (2.5), as

$$(T^{-1})^N = (STS)^N = ST^NS. \tag{2.11}$$

Therefore equation (2.10) holds.

The present map (2.1) with the restrictions (2.3) and (2.4) has four symmetries, denoted by S_0, S_1, S_2 and S_3. They are given by

$$S_0: \quad x_i' = y_i,\ y_i' = x_i \qquad i = 1, 2, 3 \tag{2.12}$$

$$S_1: \quad x_1' = y_1,\ y_1' = x_1;\ x_i' = y_j,\ y_i' = x_j \qquad i = 2, 3, j = 3, 2 \tag{2.13}$$

$$S_2: \quad x_2' = y_2,\ y_2' = x_2;\ x_i' = y_j,\ y_i' = x_j \qquad i = 1, 3, j = 3, 1 \tag{2.14}$$

$$S_3: \quad x_3' = y_3,\ y_3' = x_3;\ x_i' = y_j,\ y_i' = x_j \qquad i = 1, 2, j = 2, 1. \tag{2.15}$$

S_1, S_2 and S_3 are symmetries only if the additional symmetry of equation (2.4) is satisfied.

2.2. Commuters

We will call a product of any two symmetries, S_i and S_j, a commuter C_{ij} [9]. According to equations (2.7) and (2.8), we write

$$S_iTS_i = T^{-1} \tag{2.16}$$

$$S_jT^{-1}S_j = T. \tag{2.17}$$

Substituting equation (2.16) into equation (2.17) and using equation (2.5), we thus prove that C_{ij} commute with T,

$$C_{ij}T = TC_{ij}. \tag{2.18}$$

In terms of the four symmetries of our map (2.1), we could construct twelve commuters. Only five of them are independent:

$$C_{01} = C_{10}: \quad x'_1 = x_1,\, y'_1 = y_1;\, x'_i = x_j,\, y'_i = y_j \quad i = 2, 3, j = 3, 2 \tag{2.19}$$

$$C_{02} = C_{20}: \quad x'_2 = x_2,\, y'_2 = y_2;\, x'_i = x_j,\, y'_i = y_j \quad i = 1, 3, j = 3, 1 \tag{2.20}$$

$$C_{03} = C_{30}: \quad x'_3 = x_3,\, y'_3 = y_3;\, x'_i = x_j,\, y'_i = y_j \quad i = 1, 2, j = 2, 1 \tag{2.21}$$

$$C_{12} = C_{23} = C_{31}: \quad x'_i = x_j,\, y'_i = y_j \quad i = 1, 2, 3, j = 3, 1, 2 \tag{2.22}$$

$$C_{21} = C_{32} = C_{13}: \quad x'_i = x_j,\, y'_i = y_j \quad i = 1, 2, 3, j = 2, 3, 1. \tag{2.23}$$

In addition, the identity $I = S_iS_i$ $(i = 0, 1, 2, 3)$, of course, commutes with T.

2.3. *Invariant manifolds*

An invariant manifold of a commuter C (or any operator) is a subspace of the six-dimensional space such that, for any point $\boldsymbol{r}$ in the subspace,

$$C\boldsymbol{r} = \boldsymbol{r}. \tag{2.24}$$

The invariant manifold of the first three commuters (C_{01}, C_{02} and C_{03}) are four dimensional. They are given by

$$M_1: \quad x_2 = x_3,\, y_2 = y_3 \tag{2.25}$$

$$M_2: \quad x_3 = x_1,\, y_3 = y_1 \tag{2.26}$$

$$M_3: \quad x_1 = x_2,\, y_1 = y_2. \tag{2.27}$$

However, the invariant manifolds of the last two commuters, $C_{12}(= C_{23} = C_{31})$ and $C_{21}(= C_{32} = C_{13})$, are two dimensional. They are identical, and given by

$$M_0: \quad \begin{aligned} x_1 &= x_2 = x_3 \\ y_1 &= y_2 = y_3. \end{aligned} \tag{2.28}$$

M_0 is the intersection of M_1, M_2 and M_3.

It is obvious that if $C\boldsymbol{r} = \boldsymbol{r}$, then $C(T\boldsymbol{r}) = (T\boldsymbol{r})$. Hence, if a point $\boldsymbol{r}$ lies on an invariant manifold of a commuter C, then $(T\boldsymbol{r})$ lies also on the manifold. Therefore, if a fixed point of T^N lies on an invariant manifold M_i, then all fixed points of T^N lie on it. We call an orbit in-phase if all its elements lie on an invariant manifold M_i. Furthermore, we find a special period-doubling sequence for which all 2^n cycles $(n = 0, 1, 2, \ldots)$ lie on the invariant manifold M_0. In the next section, we will determine the period-doubling sequence for the orbits on M_0 of equation (2.28).

3. Period doubling for orbits on M_0

In this section, we numerically study period doubling for orbits on M_0 of our map (2.1), confining ourselves to the symmetry restrictions (2.3) and (2.4), with the non-linear function f_1 given by

$$f_1(x_1, x_2, x_3) = 2(Cx_1 + x_1^2) + 2E(x_2 + x_3) \tag{3.1}$$

where the term $2(Cx_1 + x_1^2)$ is the standard form of the non-linear function in the intensively studied two-dimensional Hénon map, C its parameter and E the coupling parameter. Its orbits on the manifold M_0 are $x_1 = x_2 = x_3$ as well as $y_1 = y_2 = y_3$ (cf equation (2.28)). Although our map is special (with restrictions (2.3) and (2.4)) and the periodic orbits are also special (on M_0), the universality classes numerically found in this section for our special map and special orbits cover all universality classes found by a renormalisation calculation in § 4 for the map (2.1) without those specialities.

3.1. Reduction to four-dimensional case

We introduce a coordinate transformation

$$\begin{aligned} X_1 &= \tfrac{1}{3}(x_1 + x_2 + x_3) \qquad & Y_1 &= \tfrac{1}{3}(y_1 + y_2 + y_3) \\ X_2 &= x_1 - x_2 & Y_2 &= y_1 - y_2 \\ X_3 &= x_1 - x_3 & Y_3 &= y_1 - y_3. \end{aligned} \tag{3.2}$$

In these new coordinates, the map becomes

$$\boldsymbol{X}' = -\boldsymbol{Y} + \boldsymbol{F}(\boldsymbol{X}) \qquad \boldsymbol{Y}' = \boldsymbol{X} \tag{3.3}$$

where $\boldsymbol{X} \equiv (X_1, X_2, X_3)$ and $\boldsymbol{Y} \equiv (Y_1, Y_2, Y_3)$ are 3-vectors and $\boldsymbol{F}$ is a non-linear transformation, i.e.

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{X}) = (F_1(\boldsymbol{X}), F_2(\boldsymbol{X}), F_3(\boldsymbol{X})) \\ &F_1(X_1, X_2, X_3) = 2[(C+2E)X_1 + X_1^2 + \tfrac{2}{9}(X_2^2 + X_3^2 - X_2X_3)] \\ &F_2(X_1, X_2, X_3) = 2[(C-E)X_2 + \tfrac{1}{3}(-X_2^2 + 6X_1X_2 + 2X_2X_3)] \\ &F_3(X_1, X_2, X_3) = F_2(X_1, X_3, X_2). \end{aligned} \tag{3.4}$$

For the orbits on M_0, $X_2 = Y_2 = X_3 = Y_3 = 0$. Thus X_1 and Y_1 coordinates of the in-phase orbits of the 6D map are just X_1 and Y_1 coordinates of the 2D Hénon map

$$X_1' = -Y_1 + 2(PX_1 + X_1^2) \qquad Y_1' = X_1 \tag{3.5}$$

where $P = C + 2E$. The Jacobian matrix becomes, for the orbits on M_0,

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 & 0 & 0 \\ 0 & \boldsymbol{J}_2 & 0 \\ 0 & 0 & \boldsymbol{J}_3 \end{pmatrix} \tag{3.6}$$

where $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ are 2×2 matrices,

$$\begin{aligned} \boldsymbol{J}_1 &= \begin{pmatrix} 2(C+2E) + 4X_1 & -1 \\ 1 & 0 \end{pmatrix} \\ \boldsymbol{J}_2 &= \begin{pmatrix} 2(C-E) + 4X_1 & -1 \\ 1 & 0 \end{pmatrix} \\ \boldsymbol{J}_3 &= \boldsymbol{J}_2. \end{aligned} \tag{3.7}$$

Note that $J_2 = J_3$ and $F_3(X_1, X_2, X_3) = F_2(X_1, X_3, X_2)$ in equation (3.4), both resulting from the symmetries of equations (2.3) and (2.4). Thus, bifurcation for orbits on M_0 of symmetric volume-preserving six-dimensional maps reduces to that of four-dimensional maps which we have previously studied [8].

3.2. The stability diagram

The linear stability about a fixed point of a real symplectic even-dimensional ($2N$-dimensional) map is determined by the coefficients of the reduced characteristic polynomial $Q(\rho)$ of Nth order, where ρ is the stability index

$$\rho = \lambda + 1/\lambda. \tag{3.8}$$

Here λ and $1/\lambda$ are a pair of eigenvalues of the Jacobian matrix J. The stability diagram of symplectic six-dimensional maps has already been given [11] in terms of the coefficients of $Q(\rho)$.

In fact, the stability diagram can be simply given in terms of the stability indices ρ. Note that the number of solutions of $Q(\rho) = 0$ is N, i.e. $\rho = \rho_i$, $i = 1, 2, \ldots, N$. ρ_i could be complex. The stable region of six-dimensional volume-preserving maps is just a cube, see figure 1 (note that our $\rho_1\rho_2\rho_3$ space in figure 1 is a real space and that the complex unstable region is out of our real space). Obviously, the boundaries of the stable region are given by:

(i) $\rho_i = -2$ for period-doubling bifurcation;
(ii) $\rho_i = +2$ for tangent bifurcation; and (3.9)
(iii) $\rho_i = \rho_j$, where $i \neq j$ (and entering the complex unstable region, of course), for complex bifurcation

where $i = 1, 2, \ldots, N$. These bifurcation conditions are valid not only for symplectic

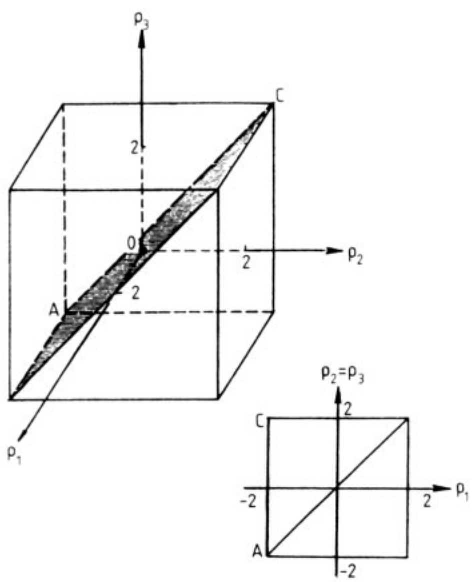

Figure 1. Stability diagram for six-dimensional volume-preserving maps in terms of the stability indices ρ_1, ρ_2 and ρ_3. Within the cube is the stable region. The shaded plane is redrawn in the inset. Note that our $\rho_1\rho_2\rho_3$ space is a real space and that the complex unstable region is out of this real space.

maps, but also for volume-preserving $2N$-dimensional real maps. Indeed, all eigenvalues of the Jacobian at an elliptic fixed point of any real $2N$-dimensional volume-preserving maps should be on the unit circle, i.e.

$$(e^{i\theta_1}, e^{-i\theta_1}; e^{i\theta_2}, e^{-i\theta_2}; \ldots; e^{i\theta_N}, e^{-i\theta_N}). \tag{3.10}$$

That is, all eigenvalues appear in reciprocal pairs, and the stability indices ρ_i of equation (3.8) are well defined, and finally the bifurcation condition (3.9) holds for volume-preserving $2N$-dimensional real maps.

The stable region for four-dimensional volume-preserving real maps is a square in the $\rho_1\rho_2$ plane, enclosed by $\rho_1=-2$, $\rho_2=-2$, $\rho_1=2$ and $\rho_2=2$. The diagonal of the square, $\rho_1=\rho_2$, is the complex bifurcation line, see the inset of figure 1. The stable region for six-dimensional volume-preserving real maps is the cube in figure 1. It is divided into two parts by the complex bifurcation plane $\rho_2=\rho_3$ (or $\rho_1=\rho_2$, or $\rho_1=\rho_3$). Period doubling occurs when we cross the plane $\rho_1=-2$, or $\rho_2=-2$, or $\rho_3=-2$.

3.3. Period-doubling sequences

For the orbits on M_0 of the map (3.3) discussed in § 3.1 it is easy now to find out the period-doubling sequence. Since $\rho_2=\rho_3$ (cf equation (3.7)), period-doubling bifurcation occurs when we cross the line AC in figure 1 if we choose $\rho_1=-2$ as the condition for period-doubling bifurcation. In other words, the inset of figure 1 is just the stability diagram. Together with the facts that our 6D map can be reduced to a 4D map (cf § 3.1), we conclude that period-doubling sequences in our symmetric 6D volume-preserving maps for the orbits on M_0 are the same sequences in symmetric 4D volume-preserving maps with $X_3=X_2$ and $Y_3=Y_2$. Therefore all results for the four-dimensional symmetric volume-preserving maps [8] are still valid for the present six-dimensional maps. These results are: stable regions bifurcate in the parameter (CE) plane; the first Feigenbaum constant $\delta_1=8.721\ldots$ for all cases; the second Feigenbaum constant $\delta_2=-4.404\ldots$, $4.000\ldots$ and -2.000 for the regular E, L and U paths, respectively; $\delta_2=-15.1\ldots$ for the exceptional L and U paths; there are three universality (E, L and U) classes; etc. In other words, for the orbits on M_0 of our symmetric 6D volume-preserving map (2.1) with the restrictions (equations (2.3) and (2.4)) on the non-linear functions and with only two parameters introduced (cf equation (3.1)), we numerically found the three universality (E, L and U) classes characterised by the second Feigenbaum constant $\delta_2=-4.404\ldots$, 4 and -2, respectively. These classes are identical to the three classes in 4D symmetric volume-preserving maps. Furthermore, these classes will be recovered by the renormalisation in the map (2.1) without the restrictions on the non-linear functions (see the next section).

4. Renormalisation for six-dimensional volume-preserving maps

In all previous sections, we have studied symmetric (i.e. with restrictions (2.3) and (2.4)) six-dimensional volume-preserving maps. In this section, we will perform a renormalisation calculation for asymmetric (i.e. without those restrictions) six-dimensional volume-preserving maps. All results given in this section (for asymmetric maps) are of course valid for symmetric maps as well.

We study the asymmetric 6D map in the DeVogelaere form [10]

$$T: \begin{cases} X_i' = -A_iY_i + F_i(\boldsymbol{X}) \\ Y_i' = (1/A_i)[X_i - F_i(\boldsymbol{X}')] \end{cases} \qquad i=1,2,3 \tag{4.1}$$

where $\boldsymbol{X} \equiv (X_1, X_2, X_3)$ and $\boldsymbol{Y} \equiv (Y_1, Y_2, Y_3)$ are 3-vectors, $A_i (i = 1, 2, 3)$ are parameters, and $\boldsymbol{F}(\boldsymbol{X}) = (F_1(\boldsymbol{X}), F_2(\boldsymbol{X}), F_3(\boldsymbol{X}))$ is a non-linear transformation,

$$F_i(\boldsymbol{X}) = B_i + \sum_{j=1}^{3} (C_{ij}X_j + D_{ij}X_j^2) + G_{i1}X_2X_3 + G_{i2}X_3X_1 + G_{i3}X_1X_2. \tag{4.2}$$

Here B_i, C_{ij}, D_{ij}, G_{ij} $(i, j = 1, 2, 3)$ are parameters. This map is truncated at quadratic terms X_1^2, X_2^2, X_3^2 and terms Y_1, Y_2, Y_3. We take this map as an approximation to the map that is fixed under the renormalisation operator. In other words, we truncate from the infinite-dimensional space of maps to a finite-dimensional space. The renormalisation of map T is

$$\boldsymbol{R}(T) = \boldsymbol{B}T^2\boldsymbol{B}^{-1} \tag{4.3}$$

where $\boldsymbol{R}$ is the period-doubling renormalisation operator composed of squaring (T^2) and rescaling $(\boldsymbol{B})$ operators. The latter is defined such that the map is a fixed point of the renormalisation operator. $\boldsymbol{R}(T)$ is also truncated to be within the finite-dimensional space.

We note that the map (4.1) is equivalent to the following Hénon-like map:

$$x_i' = -A_i y_i + 2f_i(x, u, w) \qquad y_i' = (1/A_i)x_i \tag{4.4}$$

after the DeVogelaere transformation [10]

$$\boldsymbol{x} = \boldsymbol{X} \qquad \boldsymbol{y} = \boldsymbol{Y} + \boldsymbol{f}(\boldsymbol{X}) \tag{4.5}$$

where, as before, $\boldsymbol{x} \equiv (x_1, x_2, x_3)$, $\boldsymbol{y} \equiv (y_1, y_2, y_3)$, $\boldsymbol{f}(\boldsymbol{x}) \equiv (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x}))$. Note that the Hénon-like map (4.4) identifies, when the parameters $A_1 = A_2 = A_3 = 1$, the numerically studied map (2.1), but without the restrictions of equations (2.3) and (2.4). The Hénon-like map (4.4) can be written in a second-order difference form:

$$\boldsymbol{x}_{i+1} + \boldsymbol{x}_{i-1} = 2\boldsymbol{f}(\boldsymbol{x}_i). \tag{4.6}$$

4.1. *Reduction of the number of parameters*

The map (4.1) with equation (4.2) has thirty-three parameters. However, this number of parameters can be reduced. First we notice that a fixed map under a coordinate translation is still a fixed map. Considering the equivalent map (4.6), we make translations in x_1, x_2 and x_3 such that

$$B_2 = B_3 = 0. \tag{4.7}$$

Notice that all B_1, B_2 and B_3 cannot vanish simultaneously because a complex coordinate transformation is not allowed here. Second we notice that there is neither a Y_2 term nor a Y_3 term in the X_1' equation of the DeVogelaere-like map (4.1). Hence neither has the renormalised map $\boldsymbol{R}(T)$. The expression for X_1'' in $\boldsymbol{R}(T)$ can be found easily:

$$\begin{aligned} \boldsymbol{R}(T): \qquad X_1'' &= -A_1 Y_1' + f_1(\boldsymbol{X}') \\ &= -X_1 + 2f_1(\boldsymbol{X}') \\ &= \ldots -A_2 Y_2(C_{12} + B_1 G_{13}) - A_3 Y_3(C_{13} + B_1 G_{12}) + \ldots . \end{aligned} \tag{4.8}$$

In order for $\boldsymbol{R}(T)$ to be still in the DeVogelaere form of equation (4.1), the coefficients of Y_2 and Y_3 in equation (4.8) should be zero, i.e.

$$C_{12}+B_1G_{13}=0 \tag{4.9}$$

$$C_{13}+B_1G_{12}=0. \tag{4.10}$$

Similarly, we have more constraints from expressions for X_2'' and X_3'' of $\boldsymbol{R}(T)$:

$$C_{21}+2B_1D_{21}=0 \qquad C_{23}+B_1G_{22}=0 \tag{4.11}$$

$$C_{31}+2B_1D_{31}=0 \qquad C_{32}+B_1G_{33}=0. \tag{4.12}$$

In addition, we have two more constraints since there are no constant terms in X_2'' and X_3'':

$$C_{i1}+B_1D_{i1}=0 \qquad i=2,3. \tag{4.13}$$

The solutions of these constraints (4.9)-(4.13), after a translation $X_1-B_1\rightarrow X_1$, are

$$C_{12}=C_{13}=C_{21}=C_{23}=C_{31}=C_{32}=D_{21}=D_{31}=G_{12}=G_{13}=G_{22}=G_{33}=0. \tag{4.14}$$

In other words, some terms should not appear in the non-linear functions of the map (4.2). Thus the non-linear functions can now be simplified as

$$\begin{aligned}
F_1(\boldsymbol{X})&=B+CX_1+DX_1^2+F_uX_2^2+F_wX_3^2+G_1X_2X_3\\
F_2(\boldsymbol{X})&=E_uX_2+F_1X_2^2+F_2X_3^2+G_uX_1X_2+G_2X_2X_3\\
F_3(\boldsymbol{X})&=E_wX_3+F_3X_2^2+F_4X_3^2+G_wX_1X_3+G_3X_2X_3
\end{aligned} \tag{4.15}$$

where we have introduced new parameters: B, C, D, E_u, E_w, F_u, F_w, G_u, G_w, F_j, G_i ($i=1,2,3$; $j=1,2,3,4$). Together with A_1, A_2 and A_3, we have now nineteen parameters.

4.2. Renormalisation calculation

Consider the map (4.1) with equation (4.15) as an approximation of the fixed map. We first iterate the map twice and then rescale X_1, Y_1, X_2, Y_2, X_3, Y_3 by α_1, β_1, α_2, β_2, α_3, β_3, respectively. The expression for X_1'' of the renormalised map $\boldsymbol{R}(T)$ is given by

$$\begin{aligned}
X_1''=&-Y_1\left(2\frac{\alpha_1}{\beta_1}A_1H_1\right)+[\alpha_1B(2+C+H_1)]+X_1(-1+2CH_1)\\
&+X_1^2\left(\frac{2}{\alpha_1}D(C^2+H_1)\right)+X_2^2\left(2\frac{\alpha_1}{\alpha_2^2}F_u(E_u^2+H_1)\right)\\
&+X_3^2\left(2\frac{\alpha_1}{\alpha_3^2}F_w(E_w+H_1)\right)+X_2X_3\left(2\frac{\alpha_1}{\alpha_2\alpha_3}G_1(E_uE_w+H_1)\right)
\end{aligned} \tag{4.16}$$

where $H_1=C+2BD$. Since $\boldsymbol{R}(T)$ should have the form of the original map (4.1) with (4.15), the operator $\boldsymbol{R}$ is a map from the parameter vector $\boldsymbol{P}\equiv(A_i, B, C, D, \ldots)$ onto

$P' \equiv (A_i', B', C', D', \ldots)$. Some of these mapping (from P onto P') equations are, from (4.16),

$$
\begin{aligned}
&Y_1: && A_1' = 2(\alpha_1/\beta_1)A_1 H_1 \\
&X_1^0: && B' = \alpha_1 B(2 + C + H_1) \\
&X_1: && C' = -1 + 2CH_1 \\
&X_1^2: && D' = (2/\alpha_1)D(C^2 + H_1) \\
&X_2^2: && F_u' = 2(\alpha_1/\alpha_2^2)F_u(E_u^2 + H_1) \\
&X_3^2: && F_w' = 2(\alpha_1/\alpha_3^2)F_w(E_w^2 + H_1) \\
&X_2X_3: && G_1' = 2(\alpha_1/\alpha_2\alpha_3)G_1(E_uE_w + H_1).
\end{aligned}
\tag{4.17}
$$

Similarly, from expressions for the X_2'' and X_3'' of $R(T)$, we have additional mapping equations listed in equations (A1.1) and (A1.2) of appendix 1. In all these mapping equations, equations (4.17), (A1.1) and (A1.2), we set $P' = P = P_\infty$, and then solve the equations for P_∞. P_∞ is the fixed value of the parameter P under the renormalisation. Substituting the found values of P_∞ into the non-linear functions, equation (4.15), we finally obtain the fixed map of equation (4.1) (see appendix 2 for details). The fixed maps for the 6D DeVogelaere-like map (4.1) are, after DeVogelaere transformation (4.5), as follows.

(i) Three uncoupled two-dimensional area-preserving fixed maps:

$$
x_i' = -y_i + 2(C_1 x_i + x_i^2) \qquad y_i' = x_i \qquad i = 1, 2, 3 \tag{4.18}
$$

where $C_1 = -1.2678\ldots$. In this case, the renormalisation gives $\alpha_1 = \alpha_2 = \alpha_3 = \alpha \equiv -4.0280\ldots$, $\beta_1 = \beta_2 = \beta_3 = \beta \equiv 16.5338\ldots$ and $\delta_1 = \delta \equiv 8.9474\ldots$, $\delta_2 = \delta_3 = -4.4510\ldots$, where α, β and δ are the two-dimensional scaling factors determined by the renormalisation for 2D area-preserving maps [1], and δ_1, δ_2 and δ_3 the relevant eigenvalues of the following matrices respectively,

$$
\begin{aligned}
&\left.\frac{\partial(B', C')}{\partial(B, C)}\right|_{P_\infty} \equiv \begin{bmatrix} \dfrac{\partial B'}{\partial B} & \dfrac{\partial B'}{\partial C} \\ \dfrac{\partial C'}{\partial B} & \dfrac{\partial C'}{\partial C} \end{bmatrix}_{P_\infty} \\
&\left.\frac{\partial(E_u', G_u')}{\partial(E_u, G_u)}\right|_{P_\infty} \\
&\left.\frac{\partial(E_w', G_w')}{\partial(E_w, G_w)}\right|_{P_\infty}.
\end{aligned}
\tag{4.19}
$$

The perturbation around the fixed maps, $(\partial P'/\partial P)_{P_\infty}$, contains in its diagonal the three matrices of equation (4.19) and

$$
\left.\frac{\partial A_i'}{\partial A_i}\right|_{P_\infty} = \left.\frac{\partial F_K'}{\partial F_K}\right|_{P_\infty} = \left.\frac{\partial F_j'}{\partial F_j}\right|_{P_\infty} = \left.\frac{\partial G_i'}{\partial G_i}\right|_{P_\infty} \equiv 1 \tag{4.20}
$$

where $i = 1, 2, 3$; K: u, w; $j = 1, 2, 3, 4$. The matrices of (4.19) have, in addition to δ_1,

δ_2 and δ_3, eigenvalues that arise from coordinate transformations and the truncation at terms of X_1^2, X_2^2, X_3^2, Y_1, Y_2, Y_3.

(ii) The second 6D fixed map: two two-dimensional area-preserving fixed maps coupled with a two-dimensional linear map:

$$\begin{aligned} &x_i' = -y_i + 2(C_1 x_i + x_i^2) + 2x_3^2 \qquad y_i' = x_i \qquad i = 1, 2 \\ &x_3' = -y_3 + 2C_2 x_3 \qquad y_3' = x_3 \end{aligned} \tag{4.21}$$

where C_2 could be 1 or $-\frac{1}{2}$. In this case, $\alpha_1 = \alpha_2 = \alpha$, $\alpha_3 = \pm 2.9116\ldots$ or $\pm 3.8104\ldots$ (for $C_2 = 1$ or $-\frac{1}{2}$); $\beta_1 = \beta_2 = \beta$, $\beta_3 = 2\alpha_3$ or $-\alpha_3$ (for $C_2 = 1$ or $-\frac{1}{2}$); $\delta_1 = \delta$, $\delta_2 = -4.4510\ldots$, $\delta_3 = 4$ or -2.

(iii) The third 6D fixed map: one two-dimensional area-preserving fixed map coupled with two two-dimensional linear maps:

$$\begin{aligned} &x' = -y + 2(C_1 x + x^2) + 2u^2 + 2w^2 \qquad y' = x \\ &u' = -v + 2C_2 x \qquad v' = u \\ &w' = -z + 2C_2' w \qquad z' = w \end{aligned} \tag{4.22}$$

where C_2 (or C_2') could be 1 or $-\frac{1}{2}$. In this case, $\alpha_1 = \alpha$, $\alpha_2 = \pm 2.9116\ldots$ or $\pm 3.8104\ldots$ (for $C_2 = 1$ or $-\frac{1}{2}$), $\alpha_3 = \pm 2.9116\ldots$ or $\pm 3.8104\ldots$ (for $C_2' = 1$ or $-\frac{1}{2}$); $\beta_1 = \beta$, $\beta_2 = 2\alpha_2$ or $-\alpha_2$ (for $C_2 = 1$ or $-\frac{1}{2}$), $\beta_3 = 2\alpha_3$ or $-\alpha_3$ (for $C_2' = 1$ or $-\frac{1}{2}$); $\delta_1 = \delta$, $\delta_2 = 4$ or -2 (for $C_2 = 1$ or $-\frac{1}{2}$), $\delta_3 = 4$ or -2 (for $C_2' = 1$ or $-\frac{1}{2}$).

As we can see from these fixed maps, the second Feigenbaum constant δ_2 could be one of the three values ($\delta_2 = -4.4\ldots$, 4, -2). No more δ_2 values have been found. The three δ_2 values agree with the numerical results, cf § 3.3.

5. Conclusion

We have obtained in § 3 the period-doubling sequences for orbits on M_0 of symmetric six-dimensional volume-preserving maps (equation (2.1)) with several restrictions (equations (2.3) and (2.4)) on the non-linear function, and with only two parameters (cf equation (3.1)). For this special sequence of our special map, period-doubling behaviour is the same as in symmetric four-dimensional volume-preserving maps. Also, we have performed in § 4 the renormalisation calculation for asymmetric (without those specialities) DeVogelaere-like six-dimensional maps, and found that the fixed maps are very similar to those of four-dimensional maps. In both the numerical and the renormalisation work, the three universality (E, L and U) classes have been obtained. The classes are characterised by their own second Feigenbaum constant ($\delta_2 = -4.4\ldots$, 4, -2 respectively). They are identical to those in the 4D case. However, we have not tried a numerical search for period-doubling sequences in asymmetric six-dimensional volume-preserving maps. Furthermore, the maps studied in this paper are three coupled 2D area-preserving maps, such as equations (2.1) and (4.1). A general 6D volume-preserving map is unnecessary in this (coupling) form. Finally, for 6D volume-preserving maps there are three stability indices (cf § 3.3) and thus a three-parameter family of 6D volume-preserving maps should be considered. I hope that this work can serve as a stepping stone to a complete understanding of period doubling in 6D volume-preserving maps.

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Appendix 1

Similarly to what we did in the text in order to get the mapping equations (4.17) from the expression for X_1'', we obtain the following additional mapping equations from the expression for X_2'' of the renormalisation map $\boldsymbol{R}(T)$:

$$\begin{aligned}
&Y_2: && A_2' = 2(\alpha_2/\beta_2)A_2H_2\\
&X_2: && E_u' = -1 + 2E_uH_2\\
&X_2^2: && F_1' = 2(1/\alpha_2)F_1(H_2 + E_u^2)\\
&X_3^2: && F_2' = 2(\alpha_2/\alpha_3^2)F_2(H_2 + E_w^2)\\
&X_1X_2: && G_u' = 2(1/\alpha_1)G_u(H_2 + CE_u)\\
&X_2X_3: && G_2' = 2(1/\alpha_3)G_2(H_2 + E_uE_w)
\end{aligned} \tag{A1.1}$$

where $H_2 = E_u + BG_u$. We also have, from the expression for X_3'',

$$\begin{aligned}
&Y_3: && A_3' = 2(\alpha_3/\beta_3)A_3H_3\\
&X_3: && E_w' = -1 + 2E_wH_3\\
&X_2^2: && F_3' = 2(\alpha_3/\alpha_2^2)F_3(H_3 + E_u^2)\\
&X_3^2: && F_4' = 2(1/\alpha_3)F_4(H_3 + E_w^2)\\
&X_1X_3: && G_w' = 2(1/\alpha_1)G_w(H_3 + CE_w)\\
&X_2X_3: && G_3' = 2(1/\alpha_2)G_3(H_3 + E_uE_w)
\end{aligned} \tag{A1.2}$$

where $H_3 = E_w + BG_w$.

Appendix 2

In this appendix, we describe how to obtain the fixed maps (4.18), (4.21) and (4.22) from the mapping equations (4.17), (A1.1) and (A1.2).

In order to determine the fixed values ($\boldsymbol{P}_\infty$) of the parameter under the renormalisation, we set $\boldsymbol{P}' = \boldsymbol{P} = \boldsymbol{P}_\infty$ in the mapping equations, and then solve them for $\boldsymbol{P}_\infty$. Notice that the first five equations in (4.17) and the first, second and fifth equations in (A1.1) can be solved independently from the other equations. They are equations for B, C, E_u, G_u, α_1, α_2, β_1 and β_2. They are just the same equations as those in 4D symmetric volume-preserving maps, equation (3.2) in [9]. Thus their solutions are of course just those for the 4D case, listed as E, L and U maps in table 1 of [9]. Similar discussions are valid for the first five equations in (4.17) and the first, second and fifth in (A1.2). In other words, the fixed values of parameters B, C, E_u, G_u, E_w, G_w and the scaling factors α_1, β_1, α_2, β_2, α_3, β_3 have been determined by the first five equations in (4.17) and first, second and fifth in (A1.1) and (A1.2).

The remaining equations (i.e. the sixth and seventh equations in (4.17), and the third, forth and sixth in (A1.1) and (A1.2)) can be used to determine the remaining parameters. We first note that there is no contradiction between the fifth and sixth equations in (4.17) for all cases ($E_u = E_w = C$, 1 or $-\frac{1}{2}$; or $E_u \neq E_w$). We also note that the parameters A_1, A_2, A_3, D, F_u, F_w are scales in Y_1, Y_2, Y_3, X_1, X_2, X_3 and that the parameters F_j and G_i ($i = 1, 2, 3, j = 1, 2, 3, 4$) could be any values if $E_u = E_w$, and $F_j = G_i = 0$ if $E_u \neq E_w$. These are required by the 'remaining' equations mentioned above. Hence we discuss the cases $E_u = E_w$ and $E_u \neq E_w$ separately below.

The case of $E_u = E_w$ contains three possibilities: $E_u = E_w = C$, 1 or -1. When $E_u = E_w = C$, we have $G_u = G_w = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha(\equiv -4.0280\ldots)$, $\beta_1 = \beta_2 = \beta_3 = \beta(\equiv 16.5338\ldots)$ and $\delta_1 = \delta(\equiv 8.9474\ldots)$, $\delta_2 = \delta_3 = -4.4510\ldots$. Since the parameters A_1, A_2, A_3, D, F_u, F_w and F_j, G_i are arbitrary, we set $A_1 = A_2 = A_3 = D = 1$, $F_u = F_w = G_1 = -2$, $F_1 = F_2 = 0$, $F_3 = F_4 = 3$, $G_2 = G_3 = 2$. Thus the fixed map (4.18) is obtained after a coordinate transformation $X_1 + X_2 + X_3 \to X_1$, $Y_1 + Y_2 + Y_3 \to Y_1$, $X_1 + 2X_2 + X_3 \to X_2$, $Y_1 + 2Y_2 + Y_3 \to Y_2$, $X_1 + X_2 + 2X_3 \to X_3$, $Y_1 + Y_2 + 2Y_3 \to Y_3$ and after the DeVogelaere transformation (4.5). In the cases of $E_u = E_w = 1$ or $-\frac{1}{2}$, we set $A_1 = A_2 = A_3 = D = F_u = F_w = 1$ and $F_i = G_j = 0$. Thus we obtain the fixed map (4.22) with $C_2 = C'_2 = 1$ or $-\frac{1}{2}$.

In the case of $E_u \neq E_w$, we know that $F_i = G_j = 0$. We again set the scales in X, Y, U, V, W, Z equal to 1, i.e. $A_1 = A_2 = A_3 = D = F_u = F_w = 1$. The fixed map (4.21) is obtained if either E_u or E_w is equal to C; otherwise the fixed map (4.22) is obtained.

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